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Monic Polynomials with Extremal Properties*

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We denote by $\alpha_{n,m}$ and $\beta_{n,m}$, $n \geqslant m+2$, $m \geqslant 0$, the magnitudes

$$\min_{P(x)} \max_{0 \leqslant s \leqslant m} \left\| \frac{P^{(s)}}{n(n-1)\cdots(n-s+1)} \right\|_p^p$$

extended over all the monic polynomials of degree n, where $\|\cdot\|_p = (\int_{-1}^{1} |\cdot|^p \, dx)^{1/p}$, for $p = 1, 2$, respectively. For $p = 1$ we give an asymptotically exact estimate of the values of $\alpha_{n,m}$ for $m = o(\sqrt{n})$ and we point out the polynomials which attain this estimate; for $p = 2$ we obtain

$$2^{-2n+2m+2\alpha_{n-m}} \leqslant \beta_{n,m} \leqslant 2^{-2n+3m+2\alpha_n} e^{O(m^2/n)},$$

with $1/2 < \alpha_{n-m} < \alpha_n < 3/2$.

1. INTRODUCTION

A. O. Guelfond considered in [1] the numbers $\sigma_{n,m}$, $n \geqslant m+2$, $m \geqslant 0$,

$$\sigma_{n,m} = \min_{P(x) \in \tilde{H}_n} \max_{0 \leqslant s \leqslant m} \max_{-1 \leqslant x \leqslant 1} \frac{|P^{(s)}(x)|}{n(n-1)\cdots(n-s+1)}, \tag{1}$$

where $\tilde{H}_n$ denotes the class of monic polynomials of degree n and real coefficients.

As Guelfond pointed out, the problem of finding the monic polynomials of degree n for which the minimum (1) is attained in the same way as the calculation of the value of $\sigma_{n,m}$ is a natural generalization of the well-known problem first considered and solved by Chebyshev on the monic polynomials of least supremum norm.

We shall consider the same problem for other methods of approximation, least first power approximation and least-squares approximation. The results are proved by using techniques of A. O. Guelfond adapted to the situation at hand.

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2. Least First Powers

We consider the numbers $\alpha_{n,m}$, $n \geqslant m+2$, $m \geqslant 0$,

$$\alpha_{n,m} = \min_{P(x) \in \tilde{H}_n} \max_{0 \leqslant s \leqslant m} \frac{\|P^{(s)}\|_1}{n(n-1)\cdots(n-s+1)}, \tag{2}$$

$$\text{where} \quad \|\cdot\|_1 = \int_{-1}^{1} |\cdot| \, dx.$$

We are interested in finding the monic polynomials $P(x)$ of degree n that get this minimum and the value of it. If $m=0$ it is known that $\alpha_{n,0} = 2^{1-n}$ and $P(x) = \tilde{U}_n(x)$ is the Chebyshev polynomial of the second kind normalized so that its leading coefficient is 1.

We set the polynomials

$$\begin{aligned} U_{n,0}(x) &= 2^{-n} \frac{\sin(n+1)\theta}{\sin\theta}, \qquad (x = \cos\theta),\ n \geqslant 0 \\ U_{n,k}(x) &= U_{n,k-1}(x) - \frac{n(n-1)}{4(n-k+1)(n-k)} U_{n-2,k-1}(x), \\ & \qquad 1 \leqslant k \leqslant n-2k \end{aligned} \tag{3}$$

We note that $U_{n,0}(x)$ is the polynomial $\tilde{U}_n(x)$ and $U_{n,k}(x) \in \tilde{H}_n$.

Lemma 1. *The inequality*

$$\|U_{n,k}\|_1 \leqslant 2^{-n+k+1} \mu_k(n), \qquad 1 \leqslant k \leqslant n-2k,$$

where

$$\mu_k(n) = \frac{1}{2}\left(\mu_{k-1}(n) + \frac{n(n-1)}{(n-k+1)(n-k)} \mu_{k-1}(n-2)\right),$$

holds true.

Proof. Since $\|U_{n,0}\|_1 = 2^{-n+1}$, for $k=1$, this statement follows from (3). For $k=k$ can be easily proved by induction for k.

Lemma 2. *The estimate*

$$\mu_k(n) \leqslant e^{O(k^2/n)}, \qquad 1 \leqslant k \leqslant n-2k, \tag{4}$$

holds true.

Proof. Since

$$\mu_k(n) \leqslant 2^{-k} \prod_{v=0}^{k-1} \left(1 + \frac{(n-2v)(n-2v-1)}{(n-k-v+1)(n-k-v)}\right)$$
$$\leqslant 2^{-k} \left(1 + \frac{n(n-1)}{(n-k+1)(n-k)}\right)^k,$$

(4) follows.

LEMMA 3. *The identity*

$$U_{n,k}^{(s)}(x) = \frac{n!}{(n-s)!} U_{n-s,k-s}(x), \qquad 0 \leqslant s \leqslant k, \tag{5}$$

holds true.

Proof. Since

$$U'_{n,1}(x) = 2^{-n+1} \frac{1}{2} \left\{\frac{\sin(n+1)\theta}{\sin\theta} - \frac{\sin(n-1)\theta}{\sin\theta}\right)'$$
$$= 2^{-n+1}(\cos(n \arccos x))'$$
$$= nU_{n-1,0}(x),$$

we have $U'_{n,k}(x) = nU_{n-1,k-1}(x)$, as we may readily establish by mathematical induction for k, whence there ensues (5).

Remark. We have

$$\max_{0 \leqslant s \leqslant k} \left\| \frac{U_{n,k}^{(s)}}{n(n-1)\cdots(n-s+1)} \right\|_1 \leqslant 2^{-n+k+1} e^{O(k^2/n)}, \qquad 1 \leqslant k \leqslant n-2k, \tag{6}$$

in view of Lemmas 3, 1, and 2, and $\mu_{k-s}(n-s) \leqslant \mu_k(n)$.

THEOREM 1. *If $\alpha_{n,m}$, $n \geqslant m+2$, $m \geqslant 0$, are the quantities (2), then the inequalities*

$$2^{-n+m+1} \leqslant \alpha_{n,m} \leqslant 2^{-n+m+1} e^{O(m^2/n)} \tag{7}$$

hold true.

Proof. If we consider a polynomial $P_n(x)$ such that $P_n^{(m)}(x)/(n \cdots (n-m+1))$ is the Chebyshev polynomial of the second kind of degree $n-m$ and leading coefficient 1, the first inequality holds for all $n \geqslant m+2$, $m \geqslant 0$.

With respect to the second inequality of (7), if $m=0$, it is trivial; if $1 \leqslant m \leqslant n-2m$, by (6); and if $n-2m \leqslant m \leqslant n-2$, it suffices to consider the polynomial $P_n(x)=x^n$.

Thus we come to

COROLLARY 1. *The polynomials* $U_{n,m}(x)$ *for* $m=o(\sqrt{n})$ *give us asymptotically the solution for the above problem* (2).

On the other hand, let $V_{k-1}(x)$ be a polynomial belonging to H_{k-1} (set of all polynomials whose degree not exceed the number $k-1$ with real coefficients) which minimize the expression

$$\mu_{n,k} = \min_{P(x) \in H_{k-1}} \|U_{n,k} - P\|_1.$$

We set $R_{n,k}(x) = U_{n,k}(x) - V_{k-1}(x)$.

THEOREM 2. *Among all the polynomials of degree n with the principal coefficient one,* $R_{n,k}(x)$ *is the unique polynomial which satisfies the inequality*

$$\min_{P(x) \in \tilde{H}_n} \max\left(\|P\|_1, \frac{(n-k)!\,\mu_{n,k}}{n!\,2^{-n+k+1}} \|P^{(k)}\|_1\right) \leqslant \mu_{n,k}, \qquad 1 \leqslant k \leqslant n-2k. \tag{8}$$

Proof. If the polynomial $P(x)$ satisfies (8), it satisfies

$$\frac{\|P^{(k)}\|_1}{n(n-1)\cdots(n-k+1)} \leqslant 2^{-n+k+1}. \tag{9}$$

From (9) it follows that $U_{n-k,0}(x) = P^{(k)}(x)/(n(n-1)\cdots(n-k+1))$, and with the aid of (5) we derive from it $P^{(k)}(x) = U_{n,k}^{(k)}(x)$ and hence $P(x) = R_{n,k}(x) + Q(x)$, degree of $Q(x) \leqslant k-1$. On the other hand, we know that $V_{k-1}(x)$ is unique, and therefore $Q(x)=0$.

3. LEAST SQUARES

We consider the numbers $\beta_{n,m}$, $n \geqslant m+2$, $m \geqslant 0$

$$\beta_{n,m} = \min_{P(x) \in \tilde{H}_n} \max_{0 \leqslant s \leqslant m} \left\| \frac{P^{(s)}}{n(n-1)\cdots(n-s+1)} \right\|_2^2, \tag{10}$$

where $\|\cdot\|_2 = (\int_{-1}^{1} |\cdot|^2\,dx)^{1/2}$.

Again we are interested in the value of $\beta_{n,m}$ and in the polynomials that reach such a value. For the case $m=0$ it is known that

$$P(x)=\tilde{X}_n(x)=\frac{n!}{(2n)!}\,\frac{d^n(x^2-1)^n}{dx^n}$$

is the Legendre polynomial of degree n with leading coefficient 1 and

$$\beta_{n,0}=\frac{2^{2n+1}}{2n+1}\,\frac{(n!)^4}{((2n)!)^2}=2^{-2n+2\alpha_n},$$

where for integral n, $1/2<\alpha_n<3/2$ and is defined by the last equality [3, p. 72].

We set the polynomials

$$\begin{aligned} X_{n,0}(x)&=\frac{n!}{(2n-1)!!}\,P_n(x)\\ &=\frac{n!}{(2n-1)!!}\,\frac{1}{(2n)!!}\,\frac{d^n(x^2-1)^n}{dx^n}, \qquad n\geqslant 0\\ X_{n,k}(x)&=X_{n,k-1}(x)-\frac{n(n-1)}{(2n-2k+1)(2n-2k-1)}\\ &\quad\times X_{n-2,k-1}(x), \qquad 1\leqslant k\leqslant n-2k. \end{aligned} \tag{11}$$

We observe that $X_{n,0}(x)$ is the polynomial $\tilde{X}_n(x)$ and $X_{n,k}(x)\in\tilde{H}_n$.

LEMMA 4. *The inequality*

$$\|X_{n,k}\|_2^2\leqslant 2^{-2n+k+2\alpha_n}\tau_k(n), \qquad 1\leqslant k\leqslant n-2k,$$

where

$$\tau_k(n)=\tau_{k-1}(n)+\frac{n^2(n-1)^2}{(n-k+1/2)^2\,(n-k-1/2)^2}\,\tau_{k-1}(n-2),$$

holds true.

Proof. We know that the Legendre polynomials $X_{n,0}(x)$, $n=0, 1, \ldots$ are orthogonal with respect to the weight function $p(x)=1$ on $[-1,1]$, and since $\alpha_n<\alpha_{n+1}$ we can obtain from (11)

$$\|X_{n,1}\|_2^2\leqslant 2^{-2n+2\alpha_n}\left(1+\frac{n^2(n-1)^2}{(n-1/2)^2\,(n-3/2)^2}\right).$$

Let it be correct for a value $k-1$, but no longer so for k. Note that

$$\|X_{n,k}\|_2^2 \leqslant 2\left\{\|X_{n,k-1}\|_2^2 + \frac{n^2(n-1)^2}{(2n-2k+1)^2\,(2n-2k-1)^2}\,\|X_{n-2,k-1}\|_2^2\right\},$$

where we have used the inequality $(a+b)^2 \leqslant 2(a^2+b^2)$.
Hence

$$\|X_{n,k}\|_2^2 \leqslant 2^{-2n+k+2\alpha_n}\tau_k(n),$$

where

$$\tau_k(n) = \tau_{k-1}(n) + \frac{n^2(n-1)^2}{(n-k+1/2)^2\,(n-k-1/2)^2}\,\tau_{k-1}(n-2).$$

LEMMA 5. *The estimate*

$$\tau_k(n) \leqslant 2^k e^{O(k^2/n)}, \qquad 1 \leqslant k \leqslant n-2k, \tag{12}$$

holds true.

Proof. Since

$$\begin{aligned}\tau_k(n) &\leqslant \prod_{v=0}^{k-1}\left(1+\frac{(n-2v)^2\,(n-2v-1)^2}{(n-k-v+1/2)^2\,(n-k-v-1/2)^2}\right)\\ &\leqslant \left(1+\frac{n^2(n-1)^2}{(n-k+1/2)^2\,(n-k-1/2)^2}\right)^k,\end{aligned}$$

(12) follows.

LEMMA 6. *The identity*

$$X_{n,k}^{(s)}(x) = \frac{n!}{(n-s)!}\,X_{n-s,k-s}(x), \qquad 0 \leqslant s \leqslant k, \tag{13}$$

holds true.

Proof. Since

$$\begin{aligned}X'_{n,1}(x) &= \frac{n!}{(2n-1)!!}\,(P'_n(x) - P'_{n-2}(x)) = n\,\frac{(n-1)!}{(2n-3)!!}\,P_{n-1}(x)\\ &= nX_{n-1,0}(x),\end{aligned}$$

we have $X'_{n,k}(x) = nX_{n-1,k-1}(x)$, as we may readily establish by mathematical induction for k whence there ensues (13).

Remark. Now by Lemmas 6, 4, and 5 we can obtain the inequality

$$\max_{0\leqslant s\leqslant k}\left\|\frac{X_{n,k}^{(s)}}{n(n-1)\cdots(n-s+1)}\right\|_2^2 \leqslant 2^{-2n+3k+2\alpha_n}e^{O(k^2/n)}, \qquad 1\leqslant k\leqslant n-2k. \tag{14}$$

THEOREM 3. *If* $\beta_{n,m}$, $n\geqslant m+2$, $m\geqslant 0$, *are the numbers* (10), *then the inequalities*

$$2^{-2n+2m+2\alpha_{n-m}}\leqslant \beta_{n,m}\leqslant 2^{-2n+3m+2\alpha_n}e^{O(m^2/n)} \tag{15}$$

hold true.

Proof. For every value of n and m if we consider a polynomial $P(x)$ such that $P^{(m)}(x)/(n(n-1)\cdots(n-m+1))$ is the Legendre polynomial $X_{n-m,0}(x)$ then the first inequality follows. The proof of the second inequality is similar to the second one of Theorem 1.

As before let $W_{k-1}(x)$ be the unique polynomial belonging to H_{k-1} which minimize the expression

$$\tau_{n,k}=\min_{P(x)\in H_{k-1}}\|X_{n,k}-P\|_2^2.$$

We set $S_{n,k}(x)=X_{n,k}(x)-W_{k-1}(x)$.

THEOREM 4. *Among all the polynomials of degree n with the principal coefficient one,* $S_{n,k}(x)$ *is the unique polynomial which satisfies the inequality*

$$\min_{P(x)\in \tilde{H}_n}\max\left(\|P\|_2^2, \frac{(n-k)!\,\tau_{n,k}}{n!\,2^{-2n+2m+2\alpha_{n-m}}}\|P^{(k)}\|_2^2\right) \leqslant \tau_{n,k}, \qquad 1\leqslant k\leqslant n-2k.$$

Proof. The proof runs exactly as in Theorem 2.

Note added in proof. If we consider the magnitudes

$$\tilde{\beta}_{n,m}=\min_{P(x)\in\tilde{H}_n}\max_{0\leqslant s\leqslant m}\left\|\frac{P^{(s)}}{n(n-1)\cdots(n-s+1)}\right\|_2$$

instead of (10) a similar procedure leads to the following:

$$2^{-n+m+\alpha_{n-m}}\leqslant\tilde{\beta}_{n,m}\leqslant 2^{-n+m+\alpha_n}e^{O(m^2/n)}.$$

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